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2003 J. Phys. A: Math. Gen. 36 7049

(http://iopscience.iop.org/0305-4470/36/25/310)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **36** (2003) 7049–7060 PII: S0305-4470(03)61834-2

Complete solution of the Schrödinger equation of the complex manifold CP^2

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Received 4 April 2003
Published 12 June 2003
Online at stacks.iop.org/JPhysA/36/7049

Abstract

Passing from the CP^2 Lagrangian in Kähler form, to the Hamiltonian in terms of polar coordinates, this paper gives the complete set of solutions of the corresponding Schrödinger equation in a manner that makes fully explicit the $SU(3)$ description of the energy eigenspaces. The solutions of the self-adjoint equation for the radial coordinates are derived most easily directly but also related to Jacobi polynomials.

PACS numbers: 03.65.Ge, 02.30.Ik, 02.40.Tt

1. Introduction

In this paper we give the complete set of solutions of the Schrödinger equation on the complex Riemannian manifold CP^2, and describe in detail how the energy eigenspaces carry the irreducible representations (irreps),

$$(\lambda, \lambda) \qquad \lambda \in \{0, 1, 2, \ldots\} \tag{1}$$

in standard highest weight notation, of the nonlinearly realized $SU(3)$ symmetry of the manifold.

We believe this to be of interest in its own right, but also as seen against the background of the enduring interest of CP^N manifolds in a variety of areas of theoretical physics. We next indicate briefly some aspects of this interest.

Two-dimensional CP^N field theories [1] received attention as toy models that offer a view, in a simpler context, of various aspects of Yang–Mills theories in four dimensions. They have also been studied as examples of integrable systems both in the classical and quantum settings. In addition, they are of relevance in relation to Skyrme models. Discussion of CP^N models in these contexts can be found in the book of Zakrzewski [2], which contains reference to previous and related work. Interest in such matters is still strong as some recent references indicate [3–5].

In the CP^2 case, it emerged some time ago that CP^2 can be regarded as a gravitational instanton [6] or alternatively as a gravitational plus electromagnetic instanton [7, 8]. The continued relevance of CP^N spaces to related studies of Einstein, Einstein Kähler and Einstein Sasaki spaces can be seen in papers written over the years from the days of these early papers. For example, note their widespread occurrence, especially for $N = 1, 2, 3$, within the papers [9–11]. See also [12], which reviews related material, and many other things such as the Kähler nature and Fubini–Study metric of CP^N, and [13].

The classical dynamics or ordinary quantum mechanics of the CP^N models can be regarded as field theory in a (0 + 1)-dimensional spacetime. Their gravitational use belongs to this context, as does the present paper. In particular, we wish to construct the full solution of the Schrödinger equation

$$H_q\Psi = E\Psi \tag{2}$$

of the quantum CP^2 model. The spectrum of (2) for CP^2 has been known [14] for a long time, as has the specification (1) of which $SU(3)$ representations feature in the corresponding Hilbert space, and analogous results for all CP^N. There are, further, general considerations underlying such matters, which can be indicated, following the exposition in [15]; see section 8 for a brief account. (I thank N S Manton for valuable conversations on this topic.) However, there remains the construction of all the eigenstates. We address this problem here for CP^2, $CP^2 = SU(3)/U(2)$. Using notation for isospin and hypercharge—I, I_3, Y—familiar from the application of $SU(3)$ to elementary particles, we seek a description of the eigenstates of H_q which makes manifest the (I, Y) submultiplet structure of the energy eigenspaces. That this might turn out to be non-trivial can be inferred from the fact that the CP^2 model involves two complex field variables K_i, $i = 1, 2$, and their canonically conjugate momenta, and that in it the $SU(3)$ symmetry is realized nonlinearly. Setting out from a description of the CP^2 model that reflects its Kähler structure, we introduce polar coordinates

$$(\bar{K}_1K_1 + \bar{K}_2K_2)^{1/2} = r = \tan\chi \qquad \psi, \beta, \phi \tag{3}$$

and solve (2) by separation of variables. We do this in a fashion that places the eigenstates in exact correspondence with $SU(3)$ basis states of the type $|(\lambda, \lambda)II_3Y\rangle$. The isospin factors of the wavefunctions in separated form turn out to be Wigner D-functions, familiar in the quantum theory of angular momentum [16],

$$D^I_{mI_3}(\psi, \beta, \phi) \qquad m = \tfrac{1}{2}Y \tag{4}$$

despite the fact, clarified below, that hypercharge does not enter $SU(3)$ on a footing similar to I_3. Finally, the solution of the radial equation, with independent variable changed according to $r = \tan\chi$, is seen to be of self-adjoint Sturm–Liouville type. It is very easy to solve it directly. The spectrum and the complete (I, Y) structure of the energy eigenspaces emerge in the process. The radial equation can however also be related explicitly to the equation satisfied by the Jacobi polynomials $P^n_{(2I+1),Y}(\cos 2\chi)$.

The paper begins by reviewing some facts about $SU(3)$ and about the CP^2 model as a nonlinear realization of $SU(3)$, obtaining explicit expressions for the generators of $SU(3)$ transformations: these are needed for proof that the quantum numbers that come out of the separation procedure are exactly $\mathbf{I}^2$, I_3 and Y. The paper then describes the passage from the CP^2 Lagrangian in Kähler form to the Hamitonian, shown to be essentially the Casimir operator of $SU(3)$, and hence to the Schrödinger equation, in terms of suitably defined polar coordinates. The last three sections describe its solutions and their $SU(3)$ properties.

2. Review of facts about representations of $SU(3)$

The Hermitian generators X_i, $i \in \{1, \ldots, 8\}$, of $SU(3)$ satisfy

$$[X_i, X_j] = \mathrm{i} f_{ijk} X_k \tag{5}$$

where the structure constants are specified by reference to the defining representation *def* of $SU(3)$ for which $X_i \mapsto \frac{1}{2}\lambda_i$, where the λ_i are a standard set of Gell–Mann matrices such that

$$[\lambda_i, \lambda_j] = 2\mathrm{i} f_{ijk} \lambda_k. \tag{6}$$

The quadratic Casimir oprator of $SU(3)$ is given by

$$\mathcal{C}^{(2)} = X_i X_i. \tag{7}$$

In the adjoint representation ad $X_i \mapsto F_i$, $(F_i)_{jk} = -\mathrm{i} f_{ijk}$, this takes the form

$$(\mathcal{C}^{(2)})_{jk} = f_{pqj} f_{pqk} = 3\delta_{jk}. \tag{8}$$

It is well known (for background, see, e.g., [17]) that the irreducible representations (irreps) of $SU(3)$ are classified in highest weight notation by two integers $\lambda \geqslant 0$, $\mu \geqslant 0$. For the irrep (λ, μ), we have

$$\dim(\lambda, \mu) = \tfrac{1}{2}(\lambda + 1)(\mu + 1)(\lambda + \mu + 2) \tag{9}$$

$$\mathcal{C}^{(2)}(\lambda, \mu) = \tfrac{1}{3}(\lambda^2 + \lambda\mu + \mu^2 + 3\lambda + 3\mu). \tag{10}$$

For ad $= (1, 1)$, (10) gives the eigenvalue 3, agreeing (8). Also the cubic Casimir operator, for which we do not need an explicit definition, has eigenvalue proportional to

$$(\lambda - \mu)(2\lambda + \mu + 3)(\lambda + 2\mu + 3) \tag{11}$$

for (λ, μ).

It is convenient to use definitions of $SU(3)$ generators in common use in particle physics: operators $I_\pm$, I_3, Y, $U_\pm$, $V_\pm$ which in *def* are given by

$$\begin{aligned} &I_3 \mapsto \tfrac{1}{2}\lambda_3 \qquad \sqrt{2} I_\pm = (\lambda_1 \pm \mathrm{i}\lambda_2) \qquad Y \mapsto \tfrac{\sqrt{3}}{2}\lambda_8 \\ &\sqrt{2} V_\pm = (\lambda_4 \pm \mathrm{i}\lambda_5) \qquad \sqrt{2} U_\pm = (\lambda_6 \pm \mathrm{i}\lambda_7). \end{aligned} \tag{12}$$

Also

$$\mathbf{I}^2 = I_3^2 + \tfrac{1}{2}(I_+ I_- + I_- I_+). \tag{13}$$

The states $|(\lambda, \mu); I I_3 Y\rangle$ of any (λ, μ) can then be labelled by their eigenvalues $I(I + 1)$, I_3, Y of the commuting set $\mathbf{I}^2$, I_3, Y of operators. There is a well-known algorithm (see, e.g., [18] for a simple derivation) for calculating the allowed pairs (I, Y) for states $|(\lambda, \mu); I I_3 Y\rangle$ of (λ, μ): for each pair of integers $f \geqslant 0$, $g \geqslant 0$ such that

$$0 \leqslant f \leqslant \lambda \qquad 0 \leqslant g \leqslant \mu \tag{14}$$

there is exactly one allowed (I, Y) pair given by

$$f = I + \tfrac{1}{2}Y + \tfrac{1}{3}(\lambda - \mu) \qquad g = I - \tfrac{1}{2}Y - \tfrac{1}{3}(\lambda - \mu) \tag{15}$$

so that

$$I = \tfrac{1}{2}(f + g) \qquad Y = f - g + \tfrac{2}{3}(\lambda - \mu). \tag{16}$$

The case $(I, Y) = (0, 0)$, the $U(2)$ singlet state, is of interest. Such a state occurs once in (λ, μ) if $\lambda = \mu$ and not at all otherwise. For each allowed (I, Y) pair there are $(2I + 1)$ states $|(\lambda, \mu); I I_3 Y\rangle$ of (λ, μ) with

$$-I \leqslant I_3 \leqslant I \tag{17}$$

and it can be checked that (9) follows.

Below, the subset of irrreps (λ, λ) is of major importance. For it, we have

$$\dim(\lambda, \lambda) = (\lambda + 1)^3 \tag{18}$$

$$\mathcal{C}^{(2)}(\lambda, \lambda) = \lambda(\lambda + 2) \tag{19}$$

$$\mathcal{C}^{(3)}(\lambda, \lambda) = 0. \tag{20}$$

Further, if we take $Y > 0$, then with each allowed pair (I, Y) the pair $(I, -Y)$ is also allowed. Since all the allowed values of $\left(I + \frac{1}{2}Y\right)$ are integers given by

$$0 \leqslant \left(I + \tfrac{1}{2}Y\right) \leqslant \lambda \tag{21}$$

it is then enough to specify the full set of (I, Y) values for (λ, λ).

Finally, we note that the quadratic Casimir operator of $SU(3)$ is given by

$$\mathcal{C}^{(2)} = \mathbf{I}^2 + \tfrac{3}{4}Y^2 + \tfrac{1}{2}(U_+U_- + U_-U_+ + V_+V_- + V_-V_+) \tag{22}$$

$$= I_3(I_3 + 1) + \tfrac{3}{4}Y(Y + 2) + I_-I_+ + U_-U_+ + V_-V_+. \tag{23}$$

The latter form is convenient for finding the eigenvalues of $\mathcal{C}^{(2)}$ by action on states of irreps annihilated by the operators I_+, U_+, V_+, these being the states of highest Y and the highest I_3 for that value of Y.

3. Review of CP^2 model

Following well-known lines dating back to the 1970s (see [19–21]), we set out from a three-component column vector $Z(K)$ dependent on two complex quantities K_i, $i = 1, 2$, and given by

$$Z^T = (LK_1, LK_2, L) \qquad L = L(K) = (1 + X)^{-1/2} \qquad X = (\bar{K}_1K_1 + \bar{K}_2K_2) \tag{24}$$

so that $Z^TZ = 1$. This $Z(K)$ transforms under $U \in SU(3)$ according to the law

$$Z(K) \mapsto Z(K') \qquad \text{where} \quad UZ(K) = Z(K')V(U, K). \tag{25}$$

Here $V = V(U, K)$ serves to ensure that $Z(K')_3 = L(K')$ can be chosen to be real. We need (25) first for infinitesimal

$$U = 1 + \frac{1}{2}\mathrm{i}\left(\sum_{a=1}^{3} \epsilon_a\lambda_a + \eta\lambda_8\right) \tag{26}$$

belonging to the $SU(2) \otimes U(1)$ subgroup corresponding to isospin and hypercharge transformations, and second for

$$U = 1 + \frac{1}{2}\mathrm{i}\left(\sum_{\alpha=4}^{7} \epsilon_\alpha\lambda_\alpha\right) \tag{27}$$

for those $SU(3)$ transformations that lie outside the $SU(2) \otimes U(1)$ subgroup. It is well known that the latter give rise to a nonlinear transformation of the K variables of the form

$$\delta K_i = \tfrac{1}{2}\mathrm{i}[\epsilon_i - K_i(\bar{\epsilon}_jK_j)] \tag{28}$$

where $\epsilon_1 = \epsilon_4 - \mathrm{i}\epsilon_5$, $\epsilon_2 = \epsilon_6 - \mathrm{i}\epsilon_7$.

The route from (25) to the CP^2 Lagrangian is too well known to need review. It yields the result

$$L = g_{i\bar{k}}\dot{K}_i\dot{\bar{K}}_k \tag{29}$$

where the CP^2 metric tensor is

$$g_{i\bar{k}} = (1+X)^{-1}\delta_{ik} - (1+X)^{-2}\bar{K}_i K_k \tag{30}$$

in its well-known Kähler form [12]. Proceeding classically at first, we define the canonical momenta Π_i, $\bar{\Pi}_i$, and obtain the Hamiltonian

$$H = g^{i\bar{k}}\Pi_i\bar{\Pi}_k = (1+X)[\bar{\Pi}\cdot\Pi + (\Pi\cdot K)(\bar{\Pi}\cdot\bar{K})]. \tag{31}$$

Next we apply Noether's theorem to (25). Using transformation data for the variables K_i given above, we find expressions for the generators of the infinitesimal transformations of $SU(3)$ to be

$$\begin{aligned}
&\mathrm{i}I_3 = \tfrac{1}{2}(\Pi\tau_3 K - \bar{\Pi}\tau_3\bar{K}) && \mathrm{i}Y = \Pi\cdot K - \bar{\Pi}\cdot\bar{K}\\
&\mathrm{i}I_+ = \Pi_1 K_2 - \bar{\Pi}_2\bar{K}_1 && \mathrm{i}I_- = \Pi_2 K_1 - \bar{\Pi}_1\bar{K}_2\\
&\mathrm{i}V_+ = \Pi_1 + (\bar{\Pi}\cdot\bar{K})\bar{K}_1 && -\mathrm{i}V_- = \bar{\Pi}_1 + (\Pi\cdot K)K_1\\
&\mathrm{i}U_+ = \Pi_2 + (\bar{\Pi}\cdot\bar{K})\bar{K}_2 && -\mathrm{i}U_- = \bar{\Pi}_2 + (\Pi\cdot K)K_2.
\end{aligned} \tag{32}$$

These obey the expected Poisson brackets. A direct calculation of the quadratic Casimir operator sets out from (22) and reaches the expected result $\mathcal{C}^{(2)} = H$.

In our quantum mechanical work, we prefer to deal with real variables

$$K_1 = K_4 + \mathrm{i}K_5 \qquad K_2 = K_6 + \mathrm{i}K_7. \tag{33}$$

Using Greek letters α, β etc for indices that take values in the range 4, 5, 6, 7, we find a result of the form

$$L = g_{\alpha\beta}\dot{K}_\alpha\dot{K}_\beta \tag{34}$$

where $g_{\alpha\beta}$ can be calculated from (29) and (33), but, being only an intermediate quantity in our work, is not displayed. Passing hence to the quantum mechanical Hamiltonian H_q poses an operator ordering problem when Poisson brackets relations for the canonical variables are replaced by commutation relations. One nice way to bypass these is to consider the supersymmetric extension, see, e.g., [22], of the model involving two Hermitian supercharges. Solving the operator order for these is trivial, since a simple symmetrization makes them Hermitian. Then the standard definition of the Hamiltonian in terms of these supercharges produces a correctly ordered Hamiltonian. Its non-fermionic part is then what we want [22], namely

$$H_q = g^{-1/4}\Pi_\alpha g^{1/2}g^{\alpha\beta}\Pi_\beta g^{-1/4} \tag{35}$$

where $g = \det(g_{\alpha\beta})$. Next we recognize that the scalar product of states in the Hilbert space in which H_q acts is given by

$$(\phi,\psi) = \int g^{1/2}\bar{\phi}(K)\psi(K)\prod_{\alpha=4}^{7}\mathrm{d}K_\alpha \tag{36}$$

so that the Hermitian representation

$$\Pi_\alpha = g^{-1/4}(-\mathrm{i}\partial_\alpha)g^{1/4} \qquad \partial_\alpha = \frac{\partial}{\partial K_\alpha} \tag{37}$$

should be used. This converts (35) into the form

$$H_q = -g^{-1/2}\partial_\alpha g^{1/2}g^{\alpha\beta}\partial_\beta \tag{38}$$

which is, of course, a very familiar result. See [23, p 155].

3.1. Comment on the notation in use

The column vector Z given by (24) in a slightly extended formulation [19–21] of the CP^2 model appears as the third column of the $CP^2 = SU(3)/U(2)$ coset representative $U(K) \in S(3)$

$$U(K) = \begin{pmatrix} J & LK \\ -L\bar{K} & L \end{pmatrix}. \tag{39}$$

If we had elected to employ a coset representative that uses the exponential parametrization, which of course we did not because of its intractibility, we would have written

$$U(K) = \exp \mathrm{i} \left(\sum_{\alpha=4}^{7} \lambda_\alpha K_\alpha \right) \tag{40}$$

making the notation used thereby appear natural. The coset representative (39) actually used differs from this only by some redefinition of its field variables.

4. Polar coordinates and separation of variables for CP^2

We define polar coordinates, as in [6], for the CP^2 model by setting

$$\begin{aligned} K_4 &= r \cos \tfrac{1}{2}\beta C_+ \qquad & K_5 &= -r \cos \tfrac{1}{2}\beta S_+ \\ K_6 &= r \sin \tfrac{1}{2}\beta C_- \qquad & K_7 &= r \sin \tfrac{1}{2}\beta S_- \end{aligned} \tag{41}$$

where $C_\pm = \cos \frac{1}{2}(\phi \pm \psi)$, $S_\pm = \sin \frac{1}{2}(\phi \pm \psi)$. The variable r is such that $0 \leqslant r < \infty$, and for the angular variables we have $0 \leqslant \beta \leqslant \pi, 0 \leqslant \psi \leqslant 4\pi, 0 \leqslant \phi \leqslant 2\pi$ [6].

It is easy to write the Lagrangian, (29) and (30), in terms of the real variables K_α and hence in terms of polar coordinates (41). We find, as in [6] or equivalently [7, 8],

$$\begin{aligned} L = {} & \frac{1}{(1+r^2)^2}\dot{r}^2 + \frac{r^4}{4(1+r^2)^2} \sin^2 \beta \dot{\phi}^2 \\ & + \frac{r^2}{4(1+r^2)}\dot{\beta}^2 + \frac{r^2}{4(1+r^2)^2}[(\dot{\psi}^2 + 2\cos\beta\dot{\psi}\dot{\phi} + \dot{\phi})^2]. \end{aligned} \tag{42}$$

We may read the metric tensor off (42) and hence evaluate the quantum Hamiltonian H_q of (38). For the latter purpose it is convenient to define $\chi, 0 \leqslant \chi < \frac{1}{2}\pi$ via $r = \tan\chi$. First, with coordinates r, β, ψ, ϕ, we note

$$g^{1/2}\,\mathrm{d}r = \sin^3\chi \cos\chi \sin\beta \,\mathrm{d}\chi. \tag{43}$$

Next, writing (38) in terms of the polar coordinates of (41), we get

$$\begin{aligned} H_q = {} & -\frac{1}{\sin^3\chi\cos\chi}\frac{\partial}{\partial\chi}\sin^3\chi\cos\chi\frac{\partial}{\partial\chi} - \frac{4}{\cos^2\chi}\frac{\partial^2}{\partial\psi^2} \\ & - \frac{4}{\sin^2\chi}\left[\frac{1}{\sin\beta}\frac{\partial}{\partial\beta}\sin\beta\frac{\partial}{\partial\beta} + \frac{1}{\sin^2\beta}\left(\frac{\partial^2}{\partial\phi^2} - 2\cos\beta\frac{\partial}{\partial\psi}\frac{\partial}{\partial\phi} + \frac{\partial^2}{\partial\psi^2}\right)\right]. \end{aligned} \tag{44}$$

To solve the corresponding time-independent Schrödinger equation

$$H_q\Psi = E\Psi \tag{45}$$

we separate variables using

$$\Psi = T(\chi) D^j_{m\mu}(\psi, \beta, \phi). \tag{46}$$

We do this on the basis of the quantum theory of angular momentum [16], or the representation theory of $SU(2)$, since the Wigner D-function of (46) obeys the equation

$$-\left[\frac{1}{\sin\beta}\frac{\partial}{\partial\beta}\sin\beta\frac{\partial}{\partial\beta}+\frac{1}{\sin^2\beta}\left(\frac{\partial^2}{\partial\phi^2}-2\cos\beta\frac{\partial}{\partial\psi}\frac{\partial}{\partial\phi}+\frac{\partial^2}{\partial\psi^2}\right)\right]D^j_{m\mu}(\psi,\beta,\phi)$$
$$= j(j+1)D^j_{m\mu}(\psi,\beta,\phi). \tag{47}$$

Upon writing

$$D^j_{m\mu}(\psi,\beta,\phi) = \mathrm{e}^{\mathrm{i}m\psi} d^j_{m\mu}(\beta)\,\mathrm{e}^{\mathrm{i}\mu\phi} \tag{48}$$

a well-known equation for $d^j_{m\mu}(\beta)$ follows by letting $-\mathrm{i}\partial/\partial\psi$ and $-\mathrm{i}\partial/\partial\phi$ act on the corresponding exponential factors. It is to be stressed that (47) is valid for

$$j = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots \qquad \text{and} \qquad -j \leqslant m \quad \mu \leqslant j. \tag{49}$$

It further follows that $T = T_{jm}(\chi)$ obeys the equation

$$\left[\frac{1}{\sin^3\chi\cos\chi}\frac{\mathrm{d}}{\mathrm{d}\chi}\sin^3\chi\cos\chi\frac{\mathrm{d}}{\mathrm{d}\chi}-\frac{4m^2}{\cos^2\chi}-\frac{4j(j+1)}{\sin^2\chi}\right]T = -ET. \tag{50}$$

This equation is in self-adjoint form, with weight function $\sin^3\chi\cos\chi$. Thus, if T_1 and T_2 are solutions of (50) belonging to distinct energy eigenvalues, their orthogonality relation is

$$\int_0^\pi \sin^3\chi\cos\chi\, T_1(\chi)T_2(\chi)\,\mathrm{d}\chi = 0 \tag{51}$$

just as (43) would lead one to expect, the factor $\sin\beta$, cf (43), of $g^{1/2}$ belonging to the orthogonality relation of the Wigner D-functions.

We obtain the solutions of (48) in the next section. To interpret these in relation to the $SU(3)$ structure of the spectrum of H_q, we need to identify the complete commuting set of operators which enter our method of separation of variables, and establish their connection to the labelling of states of the Hilbert space in which H_q acts. Towards this end, we return to (32), and use (33), (37) and (41) to write I_3, Y, $I_\pm$ in terms of χ, β, ψ, ϕ and the corresponding partial derivatives. The $g^{1/4}$ factors in (37) do not contribute in this process and no order of operators problem arises, and eventually one finds

$$I_3 = -\mathrm{i}\frac{\partial}{\partial\phi} \qquad Y = -2\mathrm{i}\frac{\partial}{\partial\psi} \tag{52}$$

$$\mathrm{i}I_1 = -\sin\phi\frac{\partial}{\partial\beta}+\frac{\cos\phi}{\sin\beta}\left(\frac{\partial}{\partial\psi}-\cos\beta\frac{\partial}{\partial\phi}\right) \tag{53}$$

$$\mathrm{i}I_2 = \cos\phi\frac{\partial}{\partial\beta}+\frac{\sin\phi}{\sin\beta}\left(\frac{\partial}{\partial\psi}-\cos\beta\frac{\partial}{\partial\phi}\right). \tag{54}$$

Further, we find

$$\mathbf{I}^2 = -\left[\frac{1}{\sin\beta}\frac{\partial}{\partial\beta}\sin\beta\frac{\partial}{\partial\beta}+\frac{1}{\sin^2\beta}\left(\frac{\partial^2}{\partial\phi^2}-2\cos\beta\frac{\partial}{\partial\psi}\frac{\partial}{\partial\phi}+\frac{\partial^2}{\partial\psi^2}\right)\right]. \tag{55}$$

Thus we can identify the operator that appears in (47) with the square of the isospin operator (the Casimir operator of the $SU(2)$ subgroup of $SU(3)$). It thus follows that we may rename constants appearing in (46)–(50) according to

$$m = \tfrac{1}{2}Y \qquad \mu = I_3 \qquad j = I. \tag{56}$$

We note that if we write $m\psi$ in the first factor of (48) as $m\psi = Y\psi'$, then $0 \leqslant \psi' \leqslant 2\pi$.

Our results for $\mathbf{I}$ and $\mathbf{I}^2$ coincide with results familiar in the quantum theory of angular momentum [16, p 64]. We have derived them here in the CP^2 context from first principles using Noether's theorem. Since I_3 and Y do not in general in the $SU(3)$ context bear the same relation to isospin, the appearance of Y as a label of a Wigner D-function is not something that might have been seen to be obvious a priori. In fact it will be seen to be closely related to the fact that only irreps (λ, λ) enter the analysis of CP^2. Further I_3 and Y do not feature on a similar footing in the analysis as a whole: some Y-dependence survives in (50) and is indeed essential to our discussion of the (I, Y) structure of the irreps (λ, λ) found in the Hilbert space of the CP^2 model.

We note also that (56) implies that only integral eigenvalues of Y enter our analysis for I either integral or half an odd integer, which already indicates that only irreps of $SU(3)$ of triality zero (i.e. irreps (λ, μ) such that $\lambda - \mu = 0$ mod 3) are present. The fact that $\mathcal{C}^{(3)} = 0$ implies that the qualifier mod 3 should be dropped.

5. Solution of the radial equation

We now turn to the solution of the radial equation in the form (50) that arose by writing the radial coordinate r of (41) as $r = \tan\chi$. Using also (56) we have

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}\chi^2} + (3\cot\chi - \tan\chi)\frac{\mathrm{d}}{\mathrm{d}\chi} - \frac{Y^2}{\cos^2\chi} - \frac{4I(I+1)}{\sin^2\chi}\right] T_{IY} = -E T_{IY}. \tag{57}$$

We solve this equation from first principles in this section, and relate the solutions to Jacobi polynomials in section 8. The direct method is followed because it has the advantage that it allows the $SU(3)$ structure of the energy eigenspaces to be seen most clearly.

In discussing (57) we note Y appears only via Y^2, so that for most purposes we may regard it as positive or zero. It might be thought preferable to write $|Y|$ throughout the discussion. Changing the dependent variable by means of

$$T_{IY} = \sin^{2I}\chi\cos^{Y}\chi R_{IY} \tag{58}$$

converts (57) into the form

$$\frac{\mathrm{d}^2 R}{\mathrm{d}\chi^2} + [(4I+3)\cot\chi - (2Y+1)\tan\chi]\frac{\mathrm{d}R}{\mathrm{d}\chi} + WR = 0 \tag{59}$$

where

$$W = E - (2I+Y)(2I+Y+4). \tag{60}$$

The change T to R of variables was designed to eliminate terms with $\cos^2\chi$ or $\sin^2\chi$ in their denominators. It has the further effect that (59) has as its solutions polynomials in $\cos^2\chi$

$$R_{nIY} = \sum_{r=0}^{n} a_r \cos^{2r}\chi. \tag{61}$$

Until one wishes to attend to questions of normalization, we may set $a_0 = 1$. One easily finds that (60) yields a solution of (58) for

$$\frac{a_{r+1}}{a_r} = \frac{4r(r+2I+Y+2) - W_n}{4(r+1)(Y+r+1)} \qquad r = 0, 1, 2, \ldots. \tag{62}$$

We have written $W = W_n$ for the eigenvalue W that appears in (42) when its solution is a polynomial of degree n in $\cos^2\chi$, and R_{nIY} terminates at the a_n term. This occurs when (62) implies $a_{n+1} = 0$, i.e. when

$$W_n = 4n(n+2I+Y+2). \tag{63}$$

The corresponding eigenvalue E_n in (57) then is

$$E_n = (2I + Y + 2n)(2I + Y + 2n + 4). \tag{64}$$

We now define an integer λ via

$$2\lambda = 2I + Y + 2n. \tag{65}$$

This is motivated by the fact that E_n depends on n only through λ, being given by the formula

$$E = 4\lambda(\lambda + 2). \tag{66}$$

From (19) we see that this is proportional to the eigenvalue for (λ, λ) of the quadratic Casimir operator of $SU(3)$. Since we related our Hamiltonian explicitly to this Casimir operator, this is as expected. In fact, the result is a known one; it goes back to [14], where the spectrum of all CP^N models and the irreps of $SU(N+1)$ which constitute the energy eigenspaces were found. What is thought to be new here is the full solution of the Schrödinger equation for CP^2, and the elucidation of how the (I, Y) structure of the relevant $SU(3)$ irreps emerges in a context in which the $SU(3)$ invariance is realized nonlinearly.

6. $SU(3)$ multiplet structure

For given integer λ the basis states

$$|\lambda; I I_3 Y\rangle \tag{67}$$

have wavefunctions

$$\Psi_{\lambda n I Y} = k_n T_{nIY}(\chi) \exp(\mathrm{i}m\psi) d^I_{mI_3}(\beta) \exp(\mathrm{i}I_3\phi) \qquad m = \tfrac{1}{2}Y \tag{68}$$

where k_n is a normalization constant, and

$$T_{nIY}(\chi) = \sin^{2I}\chi \cos^{2Y}\chi \, R_{nIY}(\cos\chi). \tag{69}$$

Orthogonality of wavefunctions with respect to their labels I, I_3, Y is assured by the properties of the Wigner D-functions, so that the 'radial' functions need only satisfy

$$\int_0^{\pi/2} \sin^{4I+3}\chi \cos^{2Y+1}\chi \, R_{n_1 IY} R_{n_2 IY} \, \mathrm{d}\chi = 0 \qquad \text{for} \quad n_1 \neq n_2. \tag{70}$$

The wavefunction $\Psi_{\lambda n I Y}$ has energy

$$E_\lambda = 4\lambda(\lambda + 2) \qquad \lambda = I + \tfrac{1}{2}Y + n. \tag{71}$$

Thus we have degenerate sets of states of the same energy E_λ for

$$0 \leqslant I + \tfrac{1}{2}Y \leqslant \lambda \tag{72}$$

exactly as (21) requires for (λ, λ). In fact all (I, Y) pairs, and indeed, from what has been said before, all $(I, -Y)$ compatible with (72) occur. The corresponding factor R_{NIY} of $\Psi_{\lambda n I Y}$ then is a polynomial of degree $n = \lambda - \left(I + \frac{1}{2}Y\right)$. We now give some explicit expressions for radial wavefunctions.

For $\lambda = 0$, the singlet or scalar irrep, we have only $R_{000} = 1$. For $\lambda = 1$, the octet irrep of dimension 8, we have

$$\begin{aligned} T_{010}(\chi) &= \sin^2\chi \, R_{010}(\cos\chi) = \sin^2\chi \\ T_{0\frac{1}{2}1}(\chi) &= \sin\chi\cos\chi \, R_{0\frac{1}{2}1}(\cos\chi) = \sin\chi\cos\chi \\ T_{100}(\chi) &= R_{100} = 1 - 3\cos^2\chi \end{aligned} \tag{73}$$

accounting for $3 + 2 \times 2 + 1 = 8$ states.

For $\lambda = 2$ and the irrep (2, 2) of dimension 27, we have

$$\begin{aligned}
T_{020}(\chi) &= \sin^4\chi\, R_{020}(\cos\chi) = \sin^4\chi \\
T_{0\frac{3}{2}1}(\chi) &= \sin^3\chi\cos\chi\, R_{0\frac{3}{2}1}(\cos\chi) = \sin^3\chi\cos\chi \\
T_{012}(\chi) &= \sin^2\chi\cos^2\chi\, R_{012}(\cos\chi) = \sin^2\chi\cos^2\chi \\
T_{110}(\chi) &= \sin^2\chi\, R_{110}(\cos\chi) = \sin^2\chi(1-5\cos^2\chi) \\
T_{1\frac{1}{2}1}(\chi) &= \sin\chi\cos\chi\, R_{1\frac{1}{2}1}(\cos\chi) = \sin\chi\cos\chi\left(1-\tfrac{5}{2}\cos^2\chi\right) \\
T_{200}(\chi) &= R_{200} = 1 - 8\cos^2\chi + 10\cos^4\chi.
\end{aligned} \tag{74}$$

This accounts for all $5 + 2\times 4 + 2\times 3 + 3 + 2\times 2 + 1 = 27$ states.

In addition, we note some more general results

$$R_{1IY} = 1 - \frac{2I+Y+3}{Y+1}\cos^2\chi \tag{75}$$

$$R_{2IY} = 1 - \frac{2(2I+Y+4)}{Y+1}\cos^2\chi + \left(\frac{2I+Y+4}{Y+1}\right)\left(\frac{2I+Y+5}{Y+2}\right)\cos^4\chi \tag{76}$$

and

$$R_{nIY} = \frac{n!Y!}{(2I+Y+n+1)!}\sum_{r=0}^{n}\frac{(2I+Y+n+r+1)!}{(n-r)!r!(Y+r)!}(-)^r\cos^{2r}\chi. \tag{77}$$

Since integrals like that in (70) can be evaluated in terms of gamma functions, various checks on orthogonality can be performed.

7. Solution of the radial equation in terms of Jacobi polynomials

The Jacobi polynomial $P^n_{\alpha,\beta}(x)$ [24] satisfies the differential equation

$$(1-x^2)y'' + [(\beta-\alpha) - (\alpha+\beta+2)x]y' + \lambda_n y = 0 \tag{78}$$

where $\alpha, \beta > -1$, and $-1 \leqslant x \leqslant 1$. This equation can be given in self-adjoint form with weight function and eigenvalue parameter

$$w = (1-x)^\alpha(1+x)^\beta \qquad \lambda_n = n(n+\alpha+\beta+1). \tag{79}$$

If we set

$$x = \cos 2\chi \qquad 0 \leqslant \chi \leqslant \pi/2 \tag{80}$$

then we may identify (78) with our previous radial-type equation (59) provided that we make the identifications

$$\alpha = (2I+1) \qquad \beta = Y \qquad W = 4\lambda = 4n(n+2I+Y+2) \tag{81}$$

so that, to within a constant factor, fixed below

$$R_{IY} = R_{nIY}(\cos\chi) = P^n_{(2I+1),Y}(\cos 2\chi). \tag{82}$$

The orthogonality property (70) conforms exactly to this identification, as does the result (77). The latter is however somewhat easier to work with than the standard expansion [24]

$$P^n_{\alpha,\beta}(x) = 2^{-n}\sum_{m=0}^{n}\binom{n+\alpha}{m}\binom{n+\beta}{n-m}(x-1)^{n-m}(1+x)^m. \tag{83}$$

Expansion of the term in $(x-1)^{n-m} = (-2\sin^2\chi))^{n-m}$ in powers of $\cos^2\chi$ followed by a summation over m is needed to make the connection

$$P^n_{(2I+1),Y}(\cos 2\chi) = (-)^n \binom{Y+n}{n} R_{nIY}(\cos\chi) \tag{84}$$

with (77).

Finally, we write the solutions of (2) in the form

$$\Psi_{nII_3Y}(\chi,\beta,\psi,\phi) = k_n \sin^{2I}\chi\cos^Y\chi\, P^n_{(2I+1),Y}(\cos 2\chi)\,\mathrm{e}^{\mathrm{i}Y\psi/2} d^I_{Y/2I_3}(\beta)\,\mathrm{e}^{\mathrm{i}I_3\phi} \tag{85}$$

where k_n is a new normalization constant. Their orthogonality properties have been alluded to. Their completeness can be inferred from the known completeness properties of the Jacobi polynomials and the Wigner D or d-functions.

8. A general coset space result

Here we briefly draw attention to a general result that accounts for the fact that the eigenspaces of the Schrödinger equation of CP^2 are carrier spaces of the irreps (1) of $SU(3)$.

Let G be a compact Lie group. Let $\{T^\alpha(g)\}$, for α taking values in some index set A, denote a complete system of pairwise nonequivalent unitary irreps of G. Let $d_\alpha = \dim T^\alpha(g)$. Let t^α_{ij}, $\alpha\in A$, $1\leqslant i,j\leqslant d_\alpha$, denote the matrix elements of the $T^\alpha(g)$. Then [15] the functions

$$\sqrt{d_\alpha}\, t^\alpha_{ij}(g) \qquad \alpha\in A \quad 1\leqslant i,j\leqslant d_\alpha \tag{86}$$

form a complete orthonormal basis on G for a normalized measure dg invariant on G. Hence any square integrable function on G can be decomposed into a series convergent in the mean of the form

$$f(g) = \sum_{\alpha\in A}\sum_{i,j}^{d_\alpha} c^\alpha_{ij} t^\alpha_{ij}(g). \tag{87}$$

For our purposes of this paper, the case of functions on a homogeneous space $\mathcal{M} = G/H$ of G is needed, that is functions on G that are constant on the left (or the right) cosets of $H\subset G$; we also need a few definitions. An irrep R of G is said [15] to be of class one, if its restriction to H contains the identity representation of H. If any class one irrep of G contains the identity of H exactly once, the subgroup H is termed [15] a massive subgroup of G. We now state the result we need [15]: if $f(g)$ is a function on compact G constant on the left cosets of its massive subgroup $H\subset G$, then only α of class one occurs in its expansion of type (87), and, furthermore, each class one irrep occurs exactly once.

In our work here, we have solved the Schrödinger equation (3) in terms of functions on $\mathcal{M}$, where $\mathcal{M} = CP^2$, $\quad G = SU(3)$ and $H = U(2)$ is indeed a massive subgroup of $SU(3)$. The irreps of $SU(3)$ of class one are given by (2). That the singlet state $I = Y = 0$ of the $U(2)$ subgroup of isospin rotations and hypercharge transformations occurs once in each of the class one irreps of $SU(3)$ is shown in section 2.

Acknowledgment

The research reported here is supported in part by PPARC.

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